Convergence at first and second order of some approximations of stochastic integrals

Blandine Bérard Bergery, Pierre Vallois

Convergence at first and second order of some approximations of stochastic integrals

Bérard Bergery Blandine, Vallois Pierre

IECN, Nancy-Université, CNRS, INRIA, Boulevard des Aiguillettes B.P. 239
F-54506 Vandœuvre lès Nancy

Abstract

We consider the convergence of the approximation schemes related to Itô's integral and quadratic variation, which have been developed in [13]. First, we prove that the convergence in the a.s. sense exists when the integrand is Hölder continuous and the integrator is a continuous semimartingale. Second, we investigate the second order convergence in the Brownian motion case.

Key words: stochastic integration by regularization, quadratic variation, first and second order convergence, stochastic Fubini's theorem
2000 MSC: 60F05, 60F17, 60G44, 60H05,60J65

---

1 Introduction

We consider a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, which satisfies the usual hypotheses. The notation (ucp) will stand for the convergence in probability, uniformly on the compact sets in time.

1. Let X be a real continuous $(\mathcal{F}_t)$-semimartingale. In the usual stochastic calculus, the quadratic variation and the stochastic integral with respect to X play a central role. In [10], [11] and [12], Russo and Vallois extended these notions to continuous processes. Let us briefly recall their main definitions.

Definition 1.1 *Let X be a real-valued continuous process, $(\mathcal{F}_t)$-adapted, and H be a locally integrable process. The forward integral $\int_0^t H d^- X$ is defined as*

$$\int_0^t H d^- X = \lim_{\epsilon \to 0} (ucp) \, \frac{1}{\epsilon} \int_0^t H_u \left(X_{u+\epsilon} - X_u \right) du,$$

if the limit exists. The quadratic variation is defined by

$$[X]_t = \lim_{\epsilon \to 0} (ucp) \frac{1}{\epsilon} \int_0^t \left(X_{u+\epsilon} - X_u \right)^2 du$$

if the limit exists.

In the article, X will stand for a real-valued continuous $(\mathcal{F}_t)$-semimartingale and $(H_t)_{t\geqslant 0}$ for an $(\mathcal{F}_t)$-progressively measurable process. If H is continuous, then, according to Proposition 1.1 of [10], the limits in (1.1) exist and coincide with the usual objects. In order to work with adapted processes only, we change $u+\epsilon$ into $(u+\epsilon)\wedge t$ in the above integrals. This change does not affect the limit (cf (3.3) of [13]). Consequently,

$$\int_0^t H_u dX_u = \lim_{\epsilon\to 0} (ucp)\, \frac{1}{\epsilon}\int_0^t H_u \left(X_{(u+\epsilon)\wedge t} - X_u\right) du, \tag{1.1}$$

and

$$< X >_t = \lim_{\epsilon\to 0}(ucp)\frac{1}{\epsilon}\int_0^t \left(X_{(u+\epsilon)\wedge t} - X_u\right)^2 du \tag{1.2}$$

where $\int_0^t H_u dX_u$ is the usual stochastic integral and $< X >$ is the usual quadratic variation of X.

2. First, we determine sufficient conditions under which the convergences in (1.1) and (1.2) hold in the almost sure sense. Let us mention that some results in this direction have been obtained in [2] and [5].

We say that a process Y is *locally Hölder continuous* if, for all $T > 0$, there exist $\alpha' \in]0,1]$ and a finite (random) constant C_Y such that

$$|Y_s - Y_u| \leqslant C_Y |u-s|^{\alpha'} \quad \forall u,\ s\in[0,T],\ \text{a.s.} \tag{1.3}$$

Our first result related to stochastic integral is the following.

Theorem 1.2 *If $(H_t)_{t\geqslant 0}$ is adapted and locally Hölder continuous, then*

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t H_u(X_{(u+\epsilon)\wedge t} - X_u)du = \int_0^t H_u dX_u, \tag{1.4}$$

in the sense of almost sure convergence, uniformly on the compact sets in time.

Our assumption related to (H_t) is simple but too strong as shows item 1 of Theorem 1.7 below. In [5], a general result of a.s. convergence of sequences of stochastic integrals has been given. However it cannot be applied to obtain (1.4) (see Remark 2.3).

We now consider the convergence of ϵ-integrals to the bracket of X.

Proposition 1.3 *If X is locally Hölder continuous, then*

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t (X_{(u+\epsilon)\wedge t} - X_u)^2 du = < X >_t, \tag{1.5}$$

in the sense of almost sure convergence, uniformly on the compact sets in time. Moreover, if K is a continuous process,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t K_u(X_{(u+\epsilon)\wedge t}-X_u)^2du=\int_0^t K_u d<X>_u, \tag{1.6}$$

in the sense of almost sure convergence.

3. Under the assumptions given in Theorem 1.2, we have an approximation scheme of $\int_0^{\cdot} H_s dX_s$ which converges a.s. According to Remark 2.1, the (a.s.) rate of convergence of is of order ϵ^α, when X has a finite variation and H is α-Hölder continuous. Therefore, it remains to determine the rate of convergence when X is a local martingale. This leads to introduce

$$\Delta_\epsilon(H,t)=\frac{1}{\sqrt{\epsilon}}\left[\frac{1}{\epsilon}\int_0^t H_u(X_{(u+\epsilon)\wedge t}-X_u)du-\int_0^t H_u dX_u\right],\quad t\geq 0 \tag{1.7}$$

where H is a progressively measurable and locally bounded process.

In order to study the limit in distribution of the family of processes $\big(\Delta_\epsilon(H,t),t\geq 0\big)$ as $\epsilon\to 0$, a two-steps strategy has been adopted. First, we consider the case where $X=H=B$ and B denotes the standard Brownian motion. Second, using a functional theorem of convergence we determine the limit of $\big(\Delta_\epsilon(H,t),t\geq 0\big)$. Note that in [2], some related results have been proven.

a) Suppose that $X=H=B$. In that case, using stochastic Fubini's theorem (cf relation (4.8) with $\Phi=1$) we have:

$$\Delta_\epsilon(B,t)=-W_\epsilon(t)+R^1_\epsilon(B,t),$$

where

$$W_\epsilon(t)=\int_0^t G_\epsilon(u)dB_u,\qquad G_\epsilon(u)=\frac{1}{\epsilon\sqrt{\epsilon}}\int_{(u-\epsilon)^+}^u(B_u-B_s)ds, \tag{1.8}$$

and

$$R^1_\epsilon(B,t):=\frac{1}{\sqrt{\epsilon}}\int_0^{t\wedge\epsilon}\big(\frac{s}{\epsilon}-1\big)B_s dB_s.$$

From Lemma 4.4, the process $R^1_\epsilon(B,\cdot)$ does not contribute to the limit since $R^1_\epsilon(B,\cdot)\overset{(ucp)}{\to}0$, as $\epsilon\to 0$. Therefore, the convergence of $\Delta_\epsilon(B,\cdot)$ reduces to the one of W_ϵ. We determine, more generally, in Theorem 1.4 below the limit of the pair $\big(W_\epsilon,B\big)$.

Theorem 1.4 *$(W_\epsilon(t),B_t)_{t\geqslant 0}$ converges in distribution to $(\sigma W_t,B_t)_{t\geqslant 0}$, as $\epsilon\to 0$, where W is a standard Brownian motion, independent from B, and $\sigma^2=\frac{1}{3}$.*

b) We now investigate the convergence of $(\Delta_\epsilon(H,t))_{t\geqslant 0}$. We restrict ourselves to processes H of the type $H_t=H_0+M_t+V_t$ where

1. H_0 is $\mathcal{F}_0$-measurable,

2. M_t is a Brownian martingale, i.e. $M_t = \displaystyle\int_0^t \Lambda_s dB_s$, where (Λ_t) is progressively measurable, locally bounded and is right-continuous with left-limits.

3. V is a continuous process, which is Hölder continuous with order $\alpha > 1/2$, vanishing at time 0.

Note that if $V_t = \displaystyle\int_0^t v_s ds$, where $(v_t)_{t\geqslant 0}$ is progressively measurable and locally bounded, then above condition 3 holds with $\alpha = 1$ and in that case, (H_t) is a semimartingale.

As for X, we assume that it is a Brownian martingale with representation :

$$X_t = \int_0^t \Phi(u) dB_u, \quad t \geq 0 \tag{1.9}$$

where $(\Phi(u))$ is predictable, locally bounded and right-continuous at 0.

From now on,

(W_t) denote a standard Brownian motion independent from (B_t),

and

$$\sigma := \frac{1}{\sqrt{3}}.$$

Using functional results of convergence (Proposition 3.2 and Theorem 5.1 in [4]) and Theorem 1.4, we obtain the following result.

Theorem 1.5 *1. For any $0 < t_1 < \cdots < t_n$, the random vector $(\Delta_\epsilon(H_0, t_1), \ldots, \Delta_\epsilon(H_0, t_n))$ converges in law to $\sigma H_0 \Phi(0)\big(N_0, \cdots, N_0\big)$, where N_0 is a standard Gaussian r.v, independent from $\mathcal{F}_0$.*

2. If V is a process which is locally Hölder continuous of order $\alpha > \frac{1}{2}$, then $\Delta_\epsilon(V, t)$ converges to 0 in the ucp sense as $\epsilon \to 0$.

3. If $M_t = \displaystyle\int_0^t \Lambda_s dB_s$, then the process $(\Delta_\epsilon(M, t))_{t\geqslant 0}$ converges in distribution to $(\sigma \int_0^t \Lambda_u \Phi(u) dW_u)_{t\geqslant 0}$ as $\epsilon \to 0$.

4. If $H_0 = 0$, M and V are as in points $(2) - (3)$ above, then $(\Delta_\epsilon(M + V, t))_{t\geqslant 0}$ converges in law to $(\sigma \int_0^t \Lambda_u \Phi(u) dW_u)_{t\geqslant 0}$ as $\epsilon \to 0$.

Let us discuss the assumptions of Theorem 1.5. As for item 2, the conclusion is false if $\alpha \leq 1/2$. Indeed, if we take $V_t = B_t$ then, $t \mapsto V_t$ is α-Hölder with $\alpha < 1/2$, however, as shows Theorem 1.4, the limit of $\big(\Delta_\epsilon(V, t)\big)$ equals (σW_t)

and is not null. It is likely too strong to suppose that (H_t) is a semimartingale : we can show (see Proposition 1.6 below) that $\big(\Delta_\epsilon(H,t),\ t \geq 0\big)$ converges in distribution where $H_t = h(B_t)$ and h is only supposed to be of class C^1. Note that in this case (H_t) is a Dirichlet process. However, if (H_t) is a stepwise and progressively measurable process then, we have the convergence in law of the finite dimensional distributions of $\big(\Delta_\epsilon(H,t),\ t \geq 0\big)$ but this family of processes does not converge in distribution (see Theorem 1.7 below).

Next, we consider the convergence of $\Delta_\epsilon(h(B),\cdot)$ for a large class of functions h. A function $h : \mathbb{R} \to \mathbb{R}$ is said to subexponential if there exist $C_1, C_2 > 0$ such that

$$|h(x)| \leq C_1 e^{C_2|x|}, \quad x \in \mathbb{R}. \tag{1.10}$$

Proposition 1.6 *Suppose that h is a function of class C^1 such that $h(0) = 0$ and h' is subexponential. Then, $\big(\Delta_\epsilon(h(B),t),\ t \geq 0\big)$ converges in distribution as $\epsilon \to 0$ to $\big(\sigma \int_0^t h'(B_s)\Phi(s)dW_s,\ t \geq 0\big)$.*

According to Exercise 3.13, chap. V in [8] we have :

$$h(B_t) = E\big(h(B_t)\big) + \int_0^t H(t,s)dB_s, \quad t \geq 0$$

where $H(t,s) = \varphi(t,s,B_s)$ and $\varphi(t,s,x) := E\big(h'(x + B_{t-s})\big)$.

Consequently $\big(H(t,s),\ 0 \leq s \leq t\big)$ is progressively measurable but depends on t, therefore item 3 of Theorem 1.5 cannot be applied.

c) We now focus on the case where (H_t) is a stepwise and progressively measurable process. We study the a.s. convergence of $\frac{1}{\epsilon}\int_0^{\cdot} H_u\big(X_{(u+\epsilon)\wedge t} - X_u\big)du$ towards $\int_0^{\cdot} H_u dX_u$ and the convergence in distribution of $\Delta_\epsilon(H,\cdot)$ as ϵ goes to 0.

Theorem 1.7 *Let $(a_i)_{i\in\mathbb{N}}$ be an increasing sequence of real numbers which satisfies $a_0 = 0$ and $a_n \to \infty$. Let $h, (h_i)_{i\in\mathbb{N}}$ be r.v.'s such that h_i is $\mathcal{F}_{a_i}$-measurable, h is $\mathcal{F}_0$-measurable. Let H be the progressively measurable and stepwise process:*

$$H_t = h 1\!\!1_{\{t=0\}} + \sum_{i\geqslant 0} h_i 1\!\!1_{\{t\in]a_i,a_{i+1}]\}}.$$

1. *Suppose that X is continuous, then, $\frac{1}{\epsilon}\int_0^t H_s(X_{(s+\epsilon)\wedge t} - X_s)ds$ converges almost surely to $\int_0^t H_s dX_s$, uniformly on the compact sets in time, as $\epsilon \to 0$.*

2. Suppose $h = 0$ and X is defined by (1.9). Associated with a sequence $(N_i)_{i\in\mathbb{N}}$ of i.i.d. r.v's with Gaussian law $\mathcal{N}(0,1)$, independent from B consider the piecewise and left-continuous process:

$$Z_s := \sigma\Big(h_0\Phi(0)N_0 1_{\{0<s\le a_1\}} + \sum_{i\ge 1}(h_i - h_{i-1})\Phi(a_i)N_i 1_{\{a_i<s\le a_{i+1}\}}\Big), \quad s>0$$

and $Z_0 = 0$.

Suppose that Φ is right-continuous at any point a_i. Then, for any fixed times $0 \le s_1 < \cdots < s_n$,

$$\Big((B_s,\ s\ge 0), \big(\Delta_\epsilon(H,s_1),\cdots,\Delta_\epsilon(H,s_n)\big)\Big)$$

converges in law to $\Big((B_s,\ s\ge 0), \big(Z_{s_1},\cdots,Z_{s_n}\big)\Big)$ as $\epsilon \to 0$.

A weak version of Theorem 1.7 has been given in Section 6.3 of [1].

Note that the family of processes $\big(\Delta_\epsilon(H,t),\ t\ge 0\big)$ cannot converge in the Skorokhod space to a right continuous process $\big(Z_0(t),\ t\ge 0\big)$. Indeed, according to Theorem 1.7, the map $t \in]0,a_1[\mapsto Z_0(t)$ should be constant and not null. This contradicts the fact that $Z_0(0) = 0$.

In [10], convergence in distribution of sequences of stochastic integrals are considered. We discuss in Remark 4.2 the link between Rootzen's result and ours.

4. Let us finally present our result of convergence in distribution related to the quadratic variation.

Let us consider

$$\Delta_\epsilon^{(2)}(K,t) = \frac{1}{\sqrt{\epsilon}}\left[\frac{1}{\epsilon}\int_0^t K_u(B_{(u+\epsilon)\wedge t} - B_u)^2 du - \int_0^t K_u du\right], \qquad (1.11)$$

where(K_s) is locally bounded and progressively measurable.

Proposition 1.8 *Let (K_s) be a predictable, right-continuous with left limits and locally bounded process. Then, $(\Delta_\epsilon^{(2)}(K,t))_{t\geqslant 0}$ converges in distribution to $(2\sigma\int_0^t K_u dW_u)_{t\geqslant 0}$, as $\epsilon\to 0$.*

5. Let us briefly detail the organization of the paper. Section 2 contains the proofs of the almost convergence results, i.e. Theorem 1.2 and Proposition 1.3. Then, the proof of Theorem 1.4 (resp. Propositions 1.6, 1.8 and Theorems 1.5, 1.7) is (resp. are) given in Section 3 (resp. Section 4).

In the calculations, C will stand for a generic constant (random or not). We will use several times a stochastic version of Fubini's theorem, which can be found in Section IV.5 of [8].

2 Proof of Theorem 1.2 and Proposition 1.3

We begin with the proof of Theorem 1.2 in Points **1-4** below. Then, we deduce Proposition 1.3 from Theorem 1.2 in Point **5**.

1. Let $T > 0$. We suppose that $(H_t)_{t\geqslant 0}$ is locally Hölder continuous of order α' and we study the almost sure convergence of

$$I_\epsilon(t) := \frac{1}{\epsilon}\int_0^t H_u(X_{(u+\epsilon)\wedge t} - X_u)du \text{ to } I(t) := \int_0^t H_u dX_u,$$

as $\epsilon \to 0$, uniformly on $t \in [0, T]$.

By stopping, we can suppose that $(X_t)_{0\leqslant t\leqslant T}$ and $< X >_T$ are bounded by a constant.

Let $X = X_0 + M + V$ be the canonical decomposition of X, where M is a continuous local martingale and V is an adapted process with finite variation. It is clear that $I_\epsilon(t) - I(t)$ can be decomposed as

$$\begin{aligned} I_\epsilon(t) - I(t) &= \left(\frac{1}{\epsilon}\int_0^t H_u(M_{(u+\epsilon)\wedge t} - M_u)du - \int_0^t H_u dM_u\right) \\ &\quad + \left(\frac{1}{\epsilon}\int_0^t H_u(V_{(u+\epsilon)\wedge t} - V_u)du - \int_0^t H_u dV_u\right). \end{aligned}$$

Then, Theorem 1.2 will be proved as soon as $I_\epsilon(t) - I(t)$ converges to 0, in the case where X is either a continuous local martingale or a continuous finite variation process.

We deal with the finite variation case in Point **2**. As for the martingale case, the study is divided in two steps:

1. First, we prove that there is a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ such that $I_{\epsilon_n}(t)$ converges almost surely to $I(t)$ and $\epsilon_n \to 0$ (see Point **3** below).
2. Second, we show that $I_\epsilon(t)$ converges almost surely to 0, uniformly for $t \in [0, T]$ (see Point **4** below).

2. Suppose that X has a finite variation, writing $X_{(u+\epsilon)\wedge t} - X_u = \int_u^{(u+\epsilon)\wedge t} dX_s$ and using Fubini's theorem yield to:

$$\begin{aligned} I_\epsilon(t) - I(t) &= \int_0^t \left(\frac{1}{\epsilon}\int_{(s-\epsilon)^+}^s H_u du - H_s\right) dX_s, \\ &= \int_0^t \left(\frac{1}{\epsilon}\int_{(s-\epsilon)^+}^s (H_u - H_s)du\right) dX_s - \int_0^{t\wedge\epsilon} \frac{\epsilon - s}{\epsilon} H_s dX_s. \end{aligned}$$

Using the Hölder property (1.3) (in the first integral) and the fact that H is bounded by a constant (in the second integral), we have for all $t \in [0, T]$:

$$\begin{aligned}|I_\epsilon(t)-I(t)| &\leqslant \int_0^T\left(\frac{1}{\epsilon}\int_{(s-\epsilon)^+}^s C_H|u-s|^\alpha du\right)d|X|_s+\int_0^\epsilon \frac{\epsilon-s}{\epsilon}C\,d|X|_s\\ &\leqslant C_H\epsilon^\alpha|X|_T+C(|X|_\epsilon-|X|_0).\end{aligned}\tag{2.1}$$

Consequently, $I_\epsilon(t)-I(t)$ converges almost surely to 0, as $\epsilon\to 0$, uniformly on any compact set in time.

Remark 2.1 *Note that (2.1) implies that :*

$$\sup_{0\le t\le T}\left|\int_0^t H_s\frac{X_{(s+\epsilon)\wedge t}-X_s}{\epsilon}ds-\int_0^t H_s dX_s\right|\le C\epsilon^\alpha$$

when (H_t) is α-Hölder continuous and X has finite variation.

3. In the two next points, X is a continuous martingale. We proceed as in step **2** above: observing that $X_{(u+\epsilon)\wedge t}-X_u=\int_u^{(u+\epsilon)\wedge t}dX_s$ and using Fubini's stochastic theorem come to

$$I_\epsilon(t)-I(t)=\int_0^t\left(\frac{1}{\epsilon}\int_{(s-\epsilon)^+}^s H_u du-H_s\right)dX_s. \tag{2.2}$$

Thus, $(I_\epsilon(t)-I(t))_{t\in[0,T]}$ is a continuous local martingale. Moreover, $E(<I_\epsilon-I>_t)$ is bounded since H and $<X>$ are bounded on $[0,T]$.

Let us introduce $p=\frac{2(1-\alpha)}{\alpha^2}+1$. This explicit expression of p in terms of α will be used later at the end of Point **4**. Burkholder-Davis-Gundy inequalities give:

$$E\left(\sup_{t\in[0,T]}|I_\epsilon(t)-I(t)|^p\right)\leqslant c_pE\left[\left(\int_0^T\left(\frac{1}{\epsilon}\int_{(s-\epsilon)^+}^s H_u du-H_s\right)^2 d<X>_s\right)^{\frac{p}{2}}\right].$$

The Hölder property (1.3) implies that:

$$\begin{aligned}\left|\frac{1}{\epsilon}\int_{(s-\epsilon)^+}^s H_u du-H_s\right| &\leqslant \frac{1}{\epsilon}\int_{s-\epsilon}^s|H_u-H_s|\,du\leqslant C_H\epsilon^\alpha,\quad \epsilon\leqslant s,\\ \left|\frac{1}{\epsilon}\int_{(s-\epsilon)^+}^s H_u du-H_s\right| &\leqslant \frac{1}{\epsilon}\int_0^s|H_u-H_s|\,du+\frac{\epsilon-s}{\epsilon}|H_s|\leqslant C\epsilon^\alpha,\quad s<\epsilon.\end{aligned}$$

a) Suppose that in (1.3), $C_H\le C$ for some C. Consequently,

$$\sup_{0\le s\le T}\left|\frac{1}{\epsilon}\int_{(s-\epsilon)^+}^s H_u du-H_s\right|\le C\epsilon^\alpha \tag{2.3}$$

and

$$E\left(\sup_{t\in[0,T]}|I_\epsilon(t)-I(t)|^p\right)\leqslant C\epsilon^{\alpha p}E[<X>_T]^{\frac{p}{2}}\leqslant C\epsilon^{\alpha p}.$$

Then, for any $\delta > 0$, Markov inequality leads to :

$$P\left(\sup_{t\in[0,T]} |I_\epsilon(t) - I(t)| > \delta\right) \leqslant \frac{C\epsilon^{\alpha p}}{\delta^p}. \tag{2.4}$$

Let us now define $(\epsilon_n)_{n\in\mathbb{N}^\star}$ by $\epsilon_n = n^{-\frac{2}{p\alpha}}$ for all $n > 0$. Replacing ϵ by ϵ_n in (2.4) comes to:

$$P\left(\sup_{t\in[0,T]} |I_{\epsilon_n}(t) - I(t)| > \delta\right) \leqslant \frac{C}{\delta^p} n^{-2}.$$

Since $\sum_{n=1}^{\infty} n^{-2} < \infty$, the Borel-Cantelli lemma implies that:

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |I_{\epsilon_n}(t) - I(t)| = 0, \quad a.s. \tag{2.5}$$

b) Using localization and Lemma 2.2 below we can reduce to the case where C_H is bounded by a constant. That implies (2.5).

Lemma 2.2 *Let (Y_t) be an adapted process and locally Hölder continuous with index α. Then for any $\beta \in]0, \alpha[$ there exists a continuous and adapted process $\big(Lip(Y,t)\big)$ such that*

$$|Y_u - Y_v| \leq Lip(Y,t)|u - v|^\beta, \quad u, v \in [0,t].$$

Proof of Lemma 2.2. Set :

$$Lip(Y,t) := \sup_{0\leq u,v\leq t} |\widetilde{Y}(u,v)|, \quad t \geq 0$$

where $\widetilde{Y}(u,v) := \dfrac{Y_u - Y_v}{|u-v|^\beta}$ when $u \neq v$ and 0 otherwise.

Lemma 2.2 follows from the continuity of $\widetilde{Y}$. ■

4. For all $\epsilon \in]0,1[$, let $n = n(\epsilon)$ denote the integer such that $\epsilon \in]\epsilon_{n+1}, \epsilon_n]$. Then, we decompose $I_\epsilon(t) - I(t)$ as follows:

$$I_\epsilon(t) - I(t) = (I_\epsilon(t) - I_{\epsilon_n}(t)) + (I_{\epsilon_n}(t) - I(t)).$$

(2.5) gives the almost sure convergence of $I_{\epsilon_n}(t)$ to $I(t)$, uniformly on $[0,T]$. Therefore, the a.s convergence of $I_\epsilon(t) - I(t)$ to 0 , uniformly on $[0,T]$, will be obtained as soon as $I_\epsilon(t) - I_{\epsilon_n}(t)$ goes to 0, uniformly on $[0,T]$.

From the definition of $I_\epsilon(t)$, it is easy to deduce that we have:

$$\begin{aligned} I_\epsilon(t) - I_{\epsilon_n}(t) &= \frac{1}{\epsilon}\left(\int_0^t H_u X_{(u+\epsilon)\wedge t} du - \int_0^t H_u X_{(u+\epsilon_n)\wedge t} du\right) \\ &\quad + \left(\frac{1}{\epsilon} - \frac{1}{\epsilon_n}\right)\left(\int_0^t H_u (X_{(u+\epsilon_n)\wedge t} - X_u) du\right). \end{aligned}$$

The changes of variable either $v = u + \epsilon$ or $v = u + \epsilon_n$ lead to

$$
\begin{aligned}
I_\epsilon(t) - I_{\epsilon_n}(t) &= \frac{1}{\epsilon}\int_\epsilon^{t+\epsilon} \left(H_{v-\epsilon} - H_{v-\epsilon_n}\right) X_{v\wedge t} dv \\
&\quad + \frac{\epsilon_n - \epsilon}{\epsilon\epsilon_n}\left(\int_{\epsilon_n}^t \left(H_{v-\epsilon_n} - H_v\right) X_v dv\right) + R_\epsilon(t),
\end{aligned} \tag{2.6}
$$

where we gather under the notation $R_\epsilon(t)$ all the remaining terms. Let us observe that $R_\epsilon(t)$ is the sum of terms which are of the form $\frac{1}{\epsilon}\int_a^b \dots dv$ where $|a-b| \leqslant \epsilon_n - \epsilon$ or $\left(\frac{1}{\epsilon} - \frac{1}{\epsilon_n}\right)\int_a^b \dots dv$ where $|a-b| \leqslant \epsilon_n$. Since H and X are bounded on $[0,T]$, we have

$$
|R_\epsilon(t)| \leqslant C\frac{\epsilon_n - \epsilon}{\epsilon} \quad \forall t \in [0,T]. \tag{2.7}
$$

By Hölder property (1.3), we get

$$
|H_{v-\epsilon} - H_{v-\epsilon_n}| \leqslant C(\epsilon_n - \epsilon)^\alpha, \quad |H_{v-\epsilon_n} - H_v| \leqslant C_H \epsilon_n^\alpha. \tag{2.8}
$$

Since X and H are bounded, we can deduce from (2.6), (2.7) and (2.8) that:

$$
|I_\epsilon(t) - I_{\epsilon_n}(t)| \leqslant C\Big(\frac{(\epsilon_n - \epsilon)^\alpha}{\epsilon} + \frac{(\epsilon_n - \epsilon)\epsilon_n^\alpha}{\epsilon\epsilon_n} + \frac{\epsilon - \epsilon_n}{\epsilon}\Big), \forall t \in [0,T]. \tag{2.9}
$$

Using the definition of ϵ_n, easy calculations lead to :

$$
\frac{\epsilon_n - \epsilon}{\epsilon} \leqslant Cn^{-1}, \ \frac{(\epsilon_n - \epsilon)^\alpha}{\epsilon} \leqslant Cn^{\frac{2(1-\alpha)}{p\alpha} - \alpha}, \ \frac{(\epsilon_n - \epsilon)\epsilon_n^\alpha}{\epsilon\epsilon_n} \leqslant n^{-\frac{2}{p} - 1 + \frac{2}{p\alpha}} \leqslant n^{\frac{2(1-\alpha)}{p\alpha} - \alpha}.
$$

Note that $p = \frac{2(1-\alpha)}{\alpha^2} + 1$ implies that $\frac{2(1-\alpha)}{p\alpha} - \alpha < 0$. As a result, $I_\epsilon(t) - I_{\epsilon_n}(t)$ goes to 0 a.s, uniformly on $[0,T]$, as $\epsilon \to 0$. ■

Remark 2.3 *Let (H_t) be an progressively measurable process. Suppose for simplicity that (X_t) is a local semimartingale. Let (ϵ_n) denote a sequence of decreasing positive numbers converging to 0 as $n \to \infty$. Applying Theorem 2 in [5] to (2.2) gives the a.s. convergence of $\sup_{0\le u\le T}\left|I_{\epsilon_n}(u) - I_\epsilon(u)\right|$ to 0 as $n \to \infty$, provided that*

$$
\sum_{n\ge 1}\left(\sup_{0\le u\le T}\left|H_u - \frac{1}{\epsilon_n}\int_{(u-\epsilon_n)_+}^u H_r dr\right|\right)^2 < \infty, \quad a.s. \tag{2.10}
$$

Suppose that (H_t) is locally Hölder with index α. According to (2.3), relation (2.10) holds if $\sum_{n\ge 1}\epsilon_n^\alpha < \infty$. To simplify the discussion suppose that $\epsilon_n = 1/n^\rho$, with $\rho > 0$. Obviously, the previous sum is finite if and only if $\rho\alpha > 1$.

Note that inequality (2.9) permit to prove the a.s. of $I_{\epsilon_n}(u)$ as soon as

$$\lim_{n\to\infty} \frac{(\epsilon_n-\epsilon)^\alpha}{\epsilon} = \lim_{n\to\infty} \frac{(\epsilon_n-\epsilon)\epsilon_n^\alpha}{\epsilon\epsilon_n} = \lim_{n\to\infty} \frac{\epsilon-\epsilon_n}{\epsilon} = 0.$$

Since ϵ varies in $[\epsilon_{n+1}, \epsilon_n]$, then

$$\frac{(\epsilon_n-\epsilon)^\alpha}{\epsilon} \le \frac{(\epsilon_n-\epsilon_{n+1})^\alpha}{\epsilon_{n+1}}.$$

It is easy to prove that

$$\frac{(\epsilon_n-\epsilon_{n+1})^\alpha}{\epsilon_{n+1}} \sim \frac{\rho^\alpha}{n^{(1+\rho)\alpha-\rho}}, \quad n\to\infty.$$

Therefore ρ has to be chosen such that $(1+\rho)\alpha-\rho>0$, i.e. $\rho < \frac{\alpha}{1-\alpha}$. Recall that $\rho > 1/\alpha$, then $\frac{1}{\alpha} < \frac{\alpha}{1-\alpha}$. This condition is equivalent to $\alpha > \alpha_0 := \frac{\sqrt{5}-1}{2}$. This inequality is not necessarily satisfied since it is only supposed that α belongs to $]0,1[$. Finally, our Theorem 1.2 is not a consequence of Theorem 2 of [5].

5. In this item X is supposed to be a locally Hölder continuous semimartingale. Note that replacing X by $X - X_0$ does not change (1.5). Therefore we may suppose that $X_0 = 0$.
It is clear that $\frac{1}{\epsilon}\int_0^t (X_{(u+\epsilon)\wedge t} - X_u)^2 du$ equals

$$\frac{1}{\epsilon}\left[\int_0^t X^2_{(u+\epsilon)\wedge t}du - \int_0^t X_u^2 du - 2\int_0^t X_u(X_{(u+\epsilon)\wedge t} - X_u)du\right].$$

Making the change of variable $v = u+\epsilon$ in the first integral, we easily get :

$$\frac{1}{\epsilon}\int_0^t (X_{(u+\epsilon)\wedge t} - X_u)^2 du = X_t^2 - \frac{1}{\epsilon}\int_0^{t\wedge\epsilon} X_v^2 dv - \frac{2}{\epsilon}\int_0^t X_u(X_{(u+\epsilon)\wedge t} - X_u)du.$$

Since X is continuous, $\frac{1}{\epsilon}\int_0^{t\wedge\epsilon} X_v^2 dv$ tends to 0 a.s, uniformly on $[0,T]$. Therefore, it can be deduced from Theorem 1.2 :

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t (X_{(u+\epsilon)\wedge t} - X_u)^2 du = X_t^2 - 2\int_0^t X_u dX_u \quad (a.s.).$$

Itô 's formula implies that the right-hand side of the above identity equals to $< X >_t$.

Replacing $(u+\epsilon)\wedge t$ by $u+\epsilon$ in either (1.5) or (1.6) does not change the limit. Then, identity (1.5) may be interpreted as follows : the measures $\frac{1}{\epsilon}(X_{u+\epsilon} - X_u)^2 du$ converges a.s. to the measure $d < X >_u$. That implies the a.s. convergence of $\frac{1}{\epsilon}\int_0^t K_u(X_{(u+\epsilon)\wedge t} - X_u)^2 du$ to $\int_0^t K_u d < X >_u$, for any continuous process K. ■

3 Proof of Theorem 1.4

Recall that $W_\epsilon(t)$ and $G_\epsilon(t)$ are defined by (1.8). We study the convergence in distribution of the two dimensional process $(W_\epsilon(t), B_t)$, as $\epsilon \to 0$.

First, we determine the limit in law of $W_\epsilon(t)$. In Point **1** we demonstrate preliminary results. Then, we prove the convergence of the moments of $W_\epsilon(t)$ in Point **2**. By the method of moments, the convergence in law of $W_\epsilon(t)$ for a fixed time is proven in Point **3**. We deduce the finite-dimensionnal convergence in Point **4**. Finally, Kolmogorov criterion concludes the proof in Point **5.** Then, we briefly sketch in Point **6** the proof of the joint convergence of $(W_\epsilon(t))_{t\geqslant 0}$ and $(B_t)_{t\geqslant 0}$. The approach is close to the one of $(W_\epsilon(t))_{t\geqslant 0}$.

1. We begin by calculating the moments of $W_\epsilon(t)$ and $G_\epsilon(u)$. We denote by $\stackrel{\mathcal{L}}{=}$ the equality in law.

Lemma 3.1 *$E\left[|G_\epsilon(u)|^2\right] = \frac{(u\wedge\epsilon)^3}{\epsilon^3}\sigma^2$. Moreover, for all $k \in \mathbb{N}$, there exists a constant m_k such that $E\left[|G_\epsilon(u)|^k\right] \leqslant m_k$, $\forall u \geqslant 0, \epsilon > 0$.*

Proof. First, we apply the change of variable $s = u - (u\wedge\epsilon)r$ in (1.8). Then, using the identity $(B_u - B_{u-v}; 0 \leqslant v \leqslant u) \stackrel{\mathcal{L}}{=} (B_v; 0 \leqslant v \leqslant u)$ and the scaling property of B, we get

$$G_\epsilon(u) \stackrel{\mathcal{L}}{=} \frac{(u\wedge\epsilon)\sqrt{u\wedge\epsilon}}{\epsilon\sqrt{\epsilon}} \int_0^1 B_r dr.$$

Since $\int_0^1 B_r dr \stackrel{\mathcal{L}}{=} \sigma N$, where $\sigma^2 = 1/3$ and N is a standard gaussian r.v, we obtain

$$E\left[|G_\epsilon(u)|^k\right] = \frac{(u\wedge\epsilon)^{\frac{3k}{2}}}{\epsilon^{\frac{3k}{2}}}\sigma^k E\left[|N|^k\right]. \tag{3.1}$$

Taking $k = 2$ gives $E\left[|G_\epsilon(u)|^2\right] = \frac{(t\wedge\epsilon)^3}{\epsilon^3}\sigma^2$. Using $u\wedge\epsilon \leqslant \epsilon$ and (3.1), we get $E[|G_\epsilon(u)|^k] \leqslant m_k$ with $m_k = \sigma^k E\left[|N|^k\right]$. ■

Lemma 3.2 *For all $k \geqslant 2$, there exists a constant $C(k)$ such that*

$$\forall t \geqslant 0, \quad E\left[|W_\epsilon(t)|^k\right] \leqslant C(k)\, t^{\frac{k}{2}}.$$

Moreover, for $k = 2$, we have

$$E\left[\left(W_\epsilon(u) - W_\epsilon((u-\epsilon)^+)\right)^2\right] \leqslant \sigma^2\epsilon, \quad \forall u \geqslant 0.$$

Proof. The Burkhölder-Davis-Gundy inequality and (1.8) give

$$E\left[|W_\epsilon(t)|^k\right] \leqslant c(k) E\left[\left(\int_0^t (G_\epsilon(u))^2\, du\right)^{\frac{k}{2}}\right].$$

Then, Jensen inequality implies:

$$E\left[\left(\int_0^t (G_\epsilon(u))^2\,du\right)^{\frac{k}{2}}\right] \leqslant t^{\frac{k}{2}-1} E\left[\int_0^t |G_\epsilon(u)|^k\,du\right].$$

Finaly, applying Lemma 3.1 comes to

$$E\left[|W_\epsilon(t)|^k\right] \leqslant c(k) m_k t^{\frac{k}{2}}.$$

The case $k = 2$ can be easily treated via (1.8) and Lemma 3.1:

$$\begin{aligned} E\left[\left(W_\epsilon(u) - W_\epsilon((u-\epsilon)^+)\right)^2\right] &= \int_{(u-\epsilon)^+}^u E\left[(G_\epsilon(v))^2\right] dv, \\ &= \int_{(u-\epsilon)^+}^u \sigma^2 \frac{(v\wedge\epsilon)^3}{\epsilon^3} dv \leqslant \sigma^2\epsilon. \end{aligned}$$

■

2. Let us now study the convergence of the moments of $W_\epsilon(t)$.

Proposition 3.3

$$\lim_{\epsilon\to 0} E\left[(W_\epsilon(t))^{2n}\right] = E\left[(\sigma W_t)^{2n}\right], \quad \forall n \in \mathbb{N}, t \geqslant 0. \tag{3.2}$$

Proof. a) We prove Proposition 3.3 by induction on $n \geqslant 1$.
For $n = 1$, from Lemma 3.1, we have:

$$E\left[(W_\epsilon(t))^2\right] = \int_0^t E\left[(G_\epsilon(u))^2\right] du = \int_0^t \sigma^2 \frac{(u\wedge\epsilon)^3}{\epsilon^3} du.$$

Then, $E\left[(W_\epsilon(t))^2\right]$ converges to $\sigma^2 t = E[(\sigma W_t)^2]$.
Let us suppose that (3.2) holds. First, we apply Itô's formula to $(W_\epsilon(t))^{2n+2}$. Second, taking the expectation reduces to 0 the martingale part. Finally, we get

$$E\left[(W_\epsilon(t))^{2n+2}\right] = \frac{(2n+2)(2n+1)}{2} \int_0^t E\left[(W_\epsilon(u))^{2n} (G_\epsilon(u))^2\right] du. \tag{3.3}$$

b) We admit for a while that

$$E\left[(W_\epsilon(u))^{2n} (G_\epsilon(u))^2\right] \longrightarrow \sigma^2 E\left[(\sigma W_u)^{2n}\right], \quad \forall u \geqslant 0. \tag{3.4}$$

Using Cauchy-Schwarz inequality and Lemmas 3.1, 3.2 give:

$$\begin{aligned} E\left[(W_\epsilon(u))^{2n} (G_\epsilon(u))^2\right] &\leqslant \sqrt{E\left[(W_\epsilon(u))^{4n}\right] E\left[(G_\epsilon(u))^4\right]} \\ &\leqslant \sqrt{C(4n) u^{2n} m_4} \leqslant \sqrt{C(4n) m_4}\, u^n. \end{aligned}$$

Consequently, we may apply Lebesgue's theorem to (3.3), we have

$$\begin{aligned}
\lim_{\epsilon\to 0} E\left[(W_\epsilon(t))^{2n+2}\right] &= \frac{(2n+2)(2n+1)}{2}\sigma^2 \int_0^t E\left[(\sigma W_u)^{2n}\right] du, \\
&= \frac{(2n+2)(2n+1)}{2}\sigma^{2n+2} \int_0^t u^n \frac{(2n)!}{n!\,2^n} du, \\
&= \frac{(2n+2)!}{(n+1)!\,2^{n+1}}(\sigma\sqrt{t})^{2n+2} = E\left[(\sigma W_t)^{2n+2}\right].
\end{aligned}$$

c) We have now to prove (3.4). If $u = 0$, $E\left[(W_\epsilon(0))^{2n}(G_\epsilon(0))^2\right] = 0 = \sigma^2 E\left[(\sigma W_0)^{2n}\right]$. If $u > 0$, it is clear that:

$$E\left[(W_\epsilon(u))^{2n}(G_\epsilon(u))^2\right] = E\left[\left(W_\epsilon((u-\epsilon)^+)\right)^{2n}(G_\epsilon(u))^2\right] + \xi_\epsilon(u), \qquad (3.5)$$

where

$$\xi_\epsilon(u) = E\left[\left\{(W_\epsilon(u))^{2n} - \left(W_\epsilon((u-\epsilon)^+)\right)^{2n}\right\}(G_\epsilon(u))^2\right].$$

Since $G_\epsilon(u)$ is independent from $\mathcal{F}_{(u-\epsilon)^+}$, we have

$$E\left[\left(W_\epsilon((u-\epsilon)^+)\right)^{2n}(G_\epsilon(u))^2\right] = E\left[\left(W_\epsilon((u-\epsilon)^+)\right)^{2n}\right] E\left[(G_\epsilon(u))^2\right].$$

Finally, plugging the identity above in (3.5) gives:

$$E\left[(W_\epsilon(u))^{2n}(G_\epsilon(u))^2\right] = E\left[(W_\epsilon(u))^{2n}\right] E\left[(G_\epsilon(u))^2\right] + \xi_\epsilon(u) + \tilde{\xi}_\epsilon(u),$$

where

$$\tilde{\xi}_\epsilon(u) = E\left[\left(W_\epsilon((u-\epsilon)^+)\right)^{2n} - (W_\epsilon(u))^{2n}\right] E\left[(G_\epsilon(u))^2\right].$$

Lemma 3.1 implies that $E\left[(G_\epsilon(u))^2\right]$ tends to σ^2 as $\epsilon \to 0$. The recurrence hypothesis implies that $E\left[(W_\epsilon(u))^{2n}\right]$ converges to $E\left[(\sigma W_u)^{2n}\right]$ as $\epsilon \to 0$. It remains to prove that $\xi_\epsilon(u)$ and $\tilde{\xi}_\epsilon(u)$ tend to 0 to conclude the proof.

The identity $a^{2n} - b^{2n} = (a-b)\sum_{k=0}^{2n-1} a^k b^{2n-1-k}$ implies that $\xi_\epsilon(u)$ is equal to the sum $\sum_{k=0}^{2n-1} S_k(\epsilon, u)$, where

$$S_k(\epsilon,u) = E\left[\left(W_\epsilon(u) - W_\epsilon((u-\epsilon)^+)\right)(G_\epsilon(u))^2\,(W_\epsilon(u))^k (W_\epsilon((u-\epsilon)^+))^{2n-1-k}\right].$$

Applying four times the Cauchy-Schwarz inequality yields to:

$$\begin{aligned}
|S_k(\epsilon,u)| \leqslant\ & \left[E\left(W_\epsilon(u) - W_\epsilon((u-\epsilon)^+)\right)^2\right]^{\frac{1}{2}} \left[E\left(G_\epsilon(u)\right)^8\right]^{\frac{1}{4}} \\
& \times \left[E(W_\epsilon(u))^{8k}\right]^{\frac{1}{8}} \left[E(W_\epsilon((u-\epsilon)^+))^{16n-8-8k}\right]^{\frac{1}{8}}.
\end{aligned}$$

Lemmas 3.1 and 3.2 lead to

$$|S_k(\epsilon,u)| \leqslant C(k) T^{n-\frac{1}{2}}\sqrt{\epsilon}, \quad \forall u \in [0,T].$$

Consequently, $\xi_\epsilon(u)$ tends to 0 as $\epsilon \to 0$. Using the same method, it is easy to prove that $\tilde{\xi}_\epsilon(u)$ tends to 0 as $\epsilon \to 0$. ■

3. From Proposition 3.3, it easy to deduce the convergence in law of $W_\epsilon(t)$ (t being fixed).

Proposition 3.4 *For any fixed $t \geqslant 0$, $W_\epsilon(t)$ converges in law to σW_t, as $\epsilon \to 0$.*

Remark 3.5 *Using stochastic Fubini theorem we have*

$$W_\epsilon(t) = \frac{1}{\epsilon\sqrt{\epsilon}} \int_0^t \Big(\int_0^u \big(v - (u-\epsilon)_+\big)_+ dB_v \Big) dB_u.$$

We keep notation given in [7]. Let us introduce the function f_ϵ :

$$f_\epsilon(u,v) := \frac{1}{\epsilon\sqrt{\epsilon}} \big(v - (u-\epsilon)_+\big)_+ 1_{\{0 \le v \le u \le t\}}.$$

Consequently $W_\epsilon(t) = J_2^1(f_\epsilon)$.
It is easy to prove that

$$\big(\|f_\epsilon\|_{\Delta_t^2}\big)^2 := \int_0^t \Big(\int_0^u f_\epsilon(u,v)^2 dv \Big) du = \frac{\epsilon}{12} + \frac{t-\epsilon}{3}, \quad t > \epsilon.$$

Therefore

$$\lim_{\epsilon \to 0} \|f_\epsilon\|_{\Delta_t^2} = \sigma\sqrt{t}.$$

Proposition 3 in [7] ensures that $W_\epsilon(t)$ converges in distribution to σW_t, as $\epsilon \to 0$ if and only if

$$\lim_{\epsilon \to 0} \int_{[0,t]^2} F_\epsilon(s_1,s_2)^2 ds_1 ds_2 = 0 \tag{3.6}$$

where

$$F_\epsilon(s_1,s_2) := \int_0^t \big(f_\epsilon(u,s_1) f_\epsilon(u,s_2) + f_\epsilon(s_1,u) f_\epsilon(s_2,u) \big) du.$$

Identity (3.6) can be shown by tedious calculations. This gives a new proof of Proposition 3.4.

Let us recall the method of moments.

Proposition 3.6 *Let $X, (X_n)_{n\in\mathbb{N}}$ be r.v's such that $E(|X|^k) < \infty$, $E(|X_n|^k) < \infty, \forall k, n \in \mathbb{N}$ and*

$$\overline{\lim}_{k\to\infty} \frac{[E(X^{2k})]^{\frac{1}{2k}}}{2k} < \infty. \tag{3.7}$$

If for all $k \in \mathbb{N}$, $\lim_{n\to\infty} E(X_n^k) = E(X^k)$, then X_n converges in law to X as $n \to \infty$.

Proof of Proposition 3.4. Let $t \geqslant 0$ be a fixed time. The odd moments of $W_\epsilon(t)$ are null. By Proposition 3.3, the even moments of $W_\epsilon(t)$ tends to σW_t. Since σW_t is a Gaussian r.v. with variance $\sigma\sqrt{t}$, it is easy to check that (3.7) holds. As a result, $W_\epsilon(t)$ converges in law to σW_t. ■

4. Next, we prove the finite-dimensionnal convergence.

Proposition 3.7 *Let $0 < t_1 < t_2 < \cdots < t_n$. Then, $(W_\epsilon(t_1), \ldots, W_\epsilon(t_n))$ converges in law to $(\sigma W_{t_1}, \ldots, \sigma W_{t_n})$, as $\epsilon \to 0$.*

Proof. We take $n = 2$ for simplicity. We consider $0 < t_1 < t_2$ and $\epsilon \in]0, t_1 \wedge (t_2 - t_1)[$. Since $t_1 > \epsilon$, note that $(u-\epsilon)^+ = u - \epsilon$ for $u \in [t_1, t_2]$. We begin with the decomposition:

$$W_\epsilon(t_2) = W_\epsilon(t_1) + \frac{1}{\epsilon\sqrt{\epsilon}} \int_{t_1+\epsilon}^{t_2} \left(\int_{u-\epsilon}^{u} (B_u - B_s) ds \right) dB_u + R_\epsilon^1(t_1, t_2),$$

where $R_\epsilon^1(t_1,t_2) = \frac{1}{\epsilon\sqrt{\epsilon}} \int_{t_1}^{t_1+\epsilon} \left(\int_{u-\epsilon}^{u} (B_u - B_s) ds \right) dB_u$. Let us note that $W_\epsilon(t_1)$ is independent from $\frac{1}{\epsilon\sqrt{\epsilon}} \int_{t_1+\epsilon}^{t_2} \left(\int_{u-\epsilon}^{u} (B_u - B_s) ds \right) dB_u$.

Let us introduce $B'_t = B_{t+t_1} - B_{t_1}, t \geqslant 0$. B' is a standard Brownian motion. The changes of variables $u = t_1 + v$ and $r = s - t_1$ in $\int_{t_1+\epsilon}^{t_2} \left(\int_{u-\epsilon}^{u} (B_u - B_s) ds \right) dB_u$ leads to

$$W_\epsilon(t_2) = W_\epsilon(t_1) + \Theta_\epsilon(t_1, t_2) + R_\epsilon^2(t_1,t_2) + R_\epsilon^1(t_1,t_2), \tag{3.8}$$

where

$$\begin{aligned}
\Theta_\epsilon(t_1,t_2) &= \frac{1}{\epsilon\sqrt{\epsilon}} \int_0^{t_2-t_1} \left(\int_{(v-\epsilon)^+}^{v} (B'_v - B'_r) dr \right) dB'_v, \\
R_\epsilon^2(t_1,t_2) &= \frac{1}{\epsilon\sqrt{\epsilon}} \int_0^{\epsilon} \left(\int_0^{v} (B'_v - B'_r) dr \right) dB'_v.
\end{aligned}$$

Straightforward calculation shows that $E\left[\left(R_\epsilon^1(t_1,t_2)\right)^2 \right]$ and $E\left[\left(R_\epsilon^2(t_1,t_2)\right)^2 \right]$ are bounded by $C\epsilon$. Thus, $R_\epsilon^1(t_1,t_2)$ and $R_\epsilon^1(t_1,t_2)$ converge to 0 in $L^2(\Omega)$. Proposition 3.4 gives the convergence in law of $\Theta_\epsilon(t_1,t_2)$ to $\sigma(W_{t_2} - W_{t_1})$ and the convergence in law of $W_\epsilon(t_1)$ to σW_{t_1}, as $\epsilon \to 0$.

Since $W_\epsilon(t_1)$ and $\Theta_\epsilon(t_1,t_2)$ are independent, the decomposition (3.8) implies that $(W_\epsilon(t_1), W_\epsilon(t_2) - W_\epsilon(t_1))$ converges in law to $(\sigma W_{t_1}, \sigma(W_{t_2} - W_{t_1}))$, as $\epsilon \to 0$. Proposition 3.4 follows immediately. ■

5. We end the proof of the convergence in law of the process $(W_\epsilon(t))_{t\geqslant 0}$ by showing that the family of the laws of $(W_\epsilon(t))_{t\geqslant 0}$ is tight as $\epsilon \in]0,1]$.

Lemma 3.8 *There exists a constant K such that*

$$E\left[|W_\epsilon(t) - W_\epsilon(s)|^4 \right] \leqslant K|t-s|^2, \qquad 0 \leqslant s \leqslant t, \epsilon > 0.$$

Proof. Applying Burkhölder-Davis-Gundy inequality, we obtain:

$$E\left[|W_\epsilon(t) - W_\epsilon(s)|^4 \right] \leqslant cE\left[\left(\int_s^t (G_\epsilon(u))^2 \, du \right)^2 \right] \leqslant c(t-s) \int_s^t E\left[(G_\epsilon(u))^4 \right] du.$$

Using Lemma 3.1, we get $E\left[|W_\epsilon(t)-W_\epsilon(s)|^4\right] \leqslant c\, m_4(t-s)^2$ and ends the proof (see Kolmogorov Criterion in Section XIII-1 of [8]). ■

6. To prove the joint convergence of $(W_\epsilon(t), B_t)_{t\geqslant 0}$ to $(\sigma W_t, B_t)_{t\geqslant 0}$, we mimick the approach developed in Points **1-5** above.

6.a. Convergence $(W_\epsilon(t), B_t)$ to $(\sigma W_t, B_t)$, t being fixed.
First, we prove that

$$\lim_{\epsilon\to 0} E(W_\epsilon^p(t) B_t^q) = E((\sigma W_t)^p B_t^q), \qquad p,q\in\mathbb{N}. \tag{3.9}$$

Let us note that the limit is null when either p or q is odd.

Using Itô's formula, we get

$$E\left[(W_\epsilon(t))^p B_t^q\right] = \frac{p(p-1)}{2}\alpha_1(t,\epsilon) + \frac{q(q-1)}{2}\alpha_2(t,\epsilon) + pq\alpha_3(t,\epsilon),$$

where

$$\begin{aligned}
\alpha_1(t,\epsilon) &= \int_0^t E\left[(W_\epsilon(u))^{p-2} B_u^q (G_\epsilon(u))^2\right] du,\\
\alpha_2(t,\epsilon) &= \int_0^t E\left[(W_\epsilon(u))^{p} B_u^{q-2}\right] du,\\
\alpha_3(t,\epsilon) &= \int_0^t E\left[(W_\epsilon(u))^{p-1} B_u^{q-1} G_\epsilon(u)\right] du.
\end{aligned}$$

To demonstrate (3.9), we proceed by induction on q, then by induction on p, q being fixed.
First, we apply (3.9) with $q-2$ instead of q, then we have directly:

$$\lim_{\epsilon\to 0}\alpha_2(t,\epsilon) = \int_0^t E\left[(\sigma W_u)^p\right] E\left[B_u^{q-2}\right] du.$$

As for $\alpha_1(t,\epsilon)$, we write

$$\begin{aligned}
(W_\epsilon(u))^{p-2} &= (W_\epsilon(u))^{p-2} - \big(W_\epsilon((u-\epsilon)^+)\big)^{p-2} + \big(W_\epsilon((u-\epsilon)^+)\big)^{p-2}\\
B_u^q &= B_u^q - B_{(u-\epsilon)^+}^q + B_{(u-\epsilon)^+}^q.
\end{aligned}$$

We proceed similarly with $\alpha_3(t,\epsilon)$. Reasoning as in Point **2** and using the two previous identities, we can prove:

$$\lim_{\epsilon\to 0}\alpha_1(t,\epsilon) = \sigma^2\int_0^t E\left[(\sigma W_u)^{p-2}\right] E\left[B_u^q\right] du \text{ and } \lim_{\epsilon\to 0}\alpha_3(t,\epsilon) = 0.$$

Consequently, when either p or q is odd, then $\lim_{\epsilon\to 0}\alpha_i(t,\epsilon) = 0$, $(i=1,2)$ and therefore:

$$\lim_{\epsilon\to 0} E(W_\epsilon^p(t) B_t^q) = 0 = E((\sigma W_t)^p B_t^q).$$

It remains to determine the limit in the case where p and q are even. Let us denote $p = 2p'$ and $q = 2q'$. Then we have

$$\begin{aligned}
\lim_{\epsilon \to 0} \alpha_1(t, \epsilon) &= \int_0^t \sigma^2 \frac{(p-2)!}{2^{p'-1}(p'-1)!} u^{p'-1} \sigma^{p-2} \frac{q!}{2^{q'}\,(q')!} u^{q'} du \\
&= \frac{(p-2)!\,q!}{2^{p'+q'-1}\,(p'-1)!\,(q')!\,(p'+q')} \sigma^p t^{p'+q'}, \\
\lim_{\epsilon \to 0} \alpha_2(t, \epsilon) &= \int_0^t \frac{p!}{2^{p'}\,(p')!} \sigma^p u^{p'} \frac{(q-2)!}{2^{q'-1}\,(q'-1)!} u^{q'-1} du \\
&= \frac{p!\,(q-2)!}{2^{p'+q'-1}\,(p')!\,(q'-1)!\,(p'+q')} \sigma^p t^{p'+q'}.
\end{aligned}$$

Then, it is easy to deduce

$$\lim_{\epsilon \to 0} E\left[(W_\epsilon(t))^p\, B_t^q\right] = \frac{p!}{2^{p'}\,(p')!} \sigma^p t^{p'} \frac{q!}{2^{q'}\,(q')!} t^{q'} = E\left[(\sigma W_t)^p\right] E\left[B_t^q\right].$$

Next, we use a two dimensional version of the method of moments:

Proposition 3.9 *Let $X, Y, (Y_n)_{n\in\mathbb{N}} (X_n)_{n\in\mathbb{N}}$ be r.v. whose moments are finite. Let us suppose that X and Y satisfy (3.7) and that $\forall p, q \in \mathbb{N}$, $\lim_{n\to\infty} E(X_n^p Y_n^q) = E(X^p Y^q)$. Then, (X_n, Y_n) converges in law to (X, Y) as $n \to \infty$.*

Since W_t and B_t are Gaussian r.v's, they both satisfy (3.7). Consequently, $(W_\epsilon(t), B_t)$ converges in law to $(\sigma W_t, B_t)$ as $\epsilon \to 0$.

6.b. Finite-dimensional convergence. Let $0 < t_1 < t_2$. We prove that $(W_\epsilon(t_1), W_\epsilon(t_2), B_{t_1}, B_{t_2})$ converges in law to $(\sigma W_{t_1}, \sigma W_{t_2}, B_{t_1}, B_{t_2})$. We apply decomposition (3.8) to $W_\epsilon(t_2)$.

By Point **6.a**, $(W_\epsilon(t_1), B_{t_1})$ converges in law to $(\sigma W_{t_1}, B_{t_1})$ and $(\Theta_\epsilon(t_1, t_2), B_{t_2} - B_{t_1})$ converges to $(\sigma W_{t_2} - \sigma W_{t_1}, B_{t_2} - B_{t_1})$. Since $(\Theta_\epsilon(t_1, t_2), B_{t_2} - B_{t_1})$ is independent from $(W_\epsilon(t_1), B_{t_1})$, we can conclude that $(W_\epsilon(t_1), W_\epsilon(t_2), B_{t_1}, B_{t_2})$ converges in law to $(\sigma W_{t_1}, \sigma W_{t_2}, B_{t_1}, B_{t_2})$. ■

4 Proofs of Theorems 1.5, 1.7 and Propositions 1.6, 1.8

1. Convergence in distribution of a family of stochastic integrals with respect to W_ϵ.

Denote $C\big([0,T]\big)$ the set of real valued and continuous functions defined on $[0,T]$. $C\big([0,T]\big)$ equipped with the uniform norm is a Banach space. Set $\mathcal{B}_c\big([0,T]\big)$ the Borel σ-field on $C\big([0,T]\big)$. Let $D\big([0,T]\big)$ be the space of right-continuous functions with left-limits equipped with the Skorokhod topology.

Consider a predictable, right-continuous with left-limits process (Γ_u) such that :

$$\big(\Gamma, W_\epsilon\big) \text{ converges in distribution to } \big(\Gamma, \sigma W\big), \quad \epsilon \to 0. \tag{4.1}$$

In (4.1), the pair $\big(\Gamma, W_\epsilon\big)$ is considered as an element of $D\big([0,T]\big) \times C\big([0,T]\big)$.

Proposition 4.1 *1. Let $F : \Big(\Omega \times C\big([0,T]\big),\ \sigma(B_u, u \geq 0) \otimes \mathcal{B}_c\big([0,T]\big)\Big) \to \mathbb{R}$ be a bounded an measurable map and such that for any ω, $F(\omega,\cdot)$ is continuous. Then :*

$$\lim_{\epsilon \to 0} E\big(F(\cdot, W_\epsilon)\big) = E\big(F(\cdot, \sigma W)\big). \tag{4.2}$$

2. Under (4.1), the process $\Big(\int_0^t \Gamma_u dW_\epsilon(u)\Big)_{t \geqslant 0}$ converges in distribution to $\Big(\sigma \int_0^t \Gamma_u dW_u\Big)_{t \geqslant 0}$ as $\epsilon \to 0$, where (Γ_u) is independent of (W_u).

Proof of Proposition 4.1 1) Denote $\mathcal{H}$ the set of $\sigma(B_u, u \geq 0)$-measurable and bounded r.v.'s A such that

$$\lim_{\epsilon \to 0} E\big(A\Theta(W_\epsilon)\big) = E\big(A\Theta(\sigma W)\big) = E(A)E\big(\Theta(\sigma W)\big), \tag{4.3}$$

where $\Theta :\ C\big([0,T]\big) \to \mathbb{R}$ is continuous and bounded.

It is clear that $\mathcal{H}$ is a linear vector space. Let $\big(A_n, n \geq 1\big)$ be a sequence of elements in $\mathcal{H}$ which satisfies

a) $\big(A_n, n \geq 1\big)$ converges uniformly to a bounded element A

either

b) $n \mapsto A_n$ is non-decreasing and the limit A is bounded.

Since

$$\begin{aligned} E\big(A\Theta(W_\epsilon)\big) - E\big(A\Theta(\sigma W)\big) &= E\big((A - A_n)\Theta(W_\epsilon)\big) \\ &+ E\big(A_n\Theta(W_\epsilon)\big) - E\big(A_n\Theta(\sigma W)\big) \\ &E\big((A_n - A)\Theta(\sigma W)\big) \end{aligned}$$

we have

$$\Big|E\big(A\Theta(W_\epsilon)\big) - E\big(A\Theta(\sigma W)\big)\Big| \leq CE\big(|A - A_n|\big) + \Big|E\big(A_n\Theta(W_\epsilon)\big) - E\big(A_n\Theta(\sigma W)\big)\Big|.$$

Consequently, $A \in \mathcal{H}$.

Consider the set $\mathcal{C}$ of r.v.'s of the type $f(B_{t_1}, \cdots, B_{t_n})$ where f is continuous and bounded. Theorem 1.4 implies that $\mathcal{C} \subset \mathcal{H}$. Then, (4.3) is direct consequence of Theorem T20 p 28 in [6].

According to Proposition 2.4 in [3], relations (4.3) and (4.2) are equivalent.

2) Denote $F_0 : D\big([0,T]\big) \times C\big([0,T]\big) \to \mathbb{R}$ a bounded and continuous function. Property (4.1) is a direct consequence of item 1 of Proposition 4.1 applied with :

$$F(\omega, w) := F_0\big((\Gamma_s(\omega), 0 \le s \le T), w\big), \quad w \in C\big([0,T]\big).$$

Recall that W_ϵ is a continuous martingale, which converges in distribution to σW as $\epsilon \to 0$. Then, by Proposition 3.2 of [4], W_ϵ satisfies the condition of uniform tightness. Consequently, from Theorem 5.1 of [4] and (4.1), we can deduce that for any predictable, right-continuous with left-limits process Γ, $\displaystyle\int_0^\cdot \Gamma_u dW_\epsilon(u)$ converges in distribution to $\sigma \displaystyle\int_0^\cdot \Gamma_u dW(u)$.

■

Remark 4.2 *1. The convergence in item 1 of Proposition 4.1 corresponds to the stable convergence, cf [3].*

2. According to relation (2.2), we have

$$\Delta_\epsilon(H,t) = \frac{1}{\sqrt{\epsilon}} \int_0^t \left(\frac{1}{\epsilon} \int_{(s-\epsilon)^+}^s H_u du - H_s \right) dB_s.$$

Let us apply the general result obtained in [9]. Let (ϵ_n) be a sequence of positive numbers converging to 0 as $n \to \infty$. For any $t > 0$, suppose :

$$\frac{1}{\epsilon_n} \int_0^t \left(\frac{1}{\epsilon_n} \int_{(s-\epsilon_n)^+}^s H_u du - H_s \right)^2 ds \xrightarrow{(P)} \tau(t), \quad n \to \infty \tag{4.4}$$

and

$$\sup_{0 \le r \le t} \frac{1}{\sqrt{\epsilon_n}} \left| \int_0^r \left(\frac{1}{\epsilon_n} \int_{(s-\epsilon_n)^+}^s H_u du - H_s \right) ds \right| \xrightarrow{(P)} 0, \quad n \to \infty \tag{4.5}$$

where $\big(\tau(t)\big)$ denotes a continuous process and (P) stands for the convergence in probability.

Then, from Theorem 1.2 in [9] we can deduce that

$$\big(\Delta_{\epsilon_n}(H,t),\ t \ge 0\big) \xrightarrow{(d)} \big(W(\tau(t))\ t \ge 0\big), \quad n \to \infty \tag{4.6}$$

where (W_t) is a standard Brownian motion independent from $\big(\tau(t)\big)$.

Suppose that (H_t) is of the type $H_t = H_0 + \displaystyle\int_0^t \Lambda_s dB_s + V_t$, where (Λ_t) and (V_t) satisfy the assumptions given in Theorem 1.5. Note that

$$\Big(\sigma \int_0^t \Lambda_u dW_u,\ t \ge 0\Big) \overset{(d)}{=} \Big(W\big(\sigma^2 \int_0^t \Lambda_u^2 du\big),\ t \ge 0\Big).$$

Therefore (4.6) suggests to prove (4.4) with

$$\tau(t) := \sigma^2 \int_0^t \Lambda_u^2 du, \quad t \geq 0.$$

We have tried without any success to directly prove (4.4) and (4.5). In the particular case $H_t = B_t$, the calculations are tractable. Theorem 1.2 in [9] may be applied : $\big(\Delta_{\epsilon_n}(B,t),\ t \geq 0\big)$ converges in distribution to $(\sigma W_t,\ t \geq 0)$, as $n \to \infty$. However, this result is not sufficient to have the convergence of $\big(\Delta_{\epsilon_n}(H,t),\ t \geq 0\big)$ since we need the convergence of $\big(\Delta_{\epsilon_n}(B,t), B_t\big)$ and the convergence of the previous pair of processes is not given by Theorem 1.2 in [9].

2. Some preliminary results related to the proof of Theorem 1.5.

Lemma 4.3 *Let $\big(\xi_\epsilon(t), t \geq 0\big)$ be a family of processes. Suppose there exists a increasing sequence $(T_n)_{n\geq 1}$ of random times such that $T_n \uparrow \infty$ as $n \to \infty$ and for any $n \geq 1$, $\big(\xi_\epsilon(t \wedge T_n), t \geq 0\big)$ converges in the ucp sense to 0, as $\epsilon \to 0$. Then $\big(\xi_\epsilon(t), t \geq 0\big)$ converges in the ucp sense to 0, as $\epsilon \to 0$, i.e. for any $T > 0$, $\sup_{0\leq s\leq T} |\xi_\epsilon(s)| \to 0$ in probability as $\epsilon \to 0$.*

Lemma 4.4 *Denote (K_t) an progressively measurable process which is right-continuous at 0, $K_0 = 0$ and locally bounded. Set :*

$$R_\epsilon^1(K,t) := \frac{1}{\sqrt{\epsilon}} \int_0^{t\wedge\epsilon} K_s\big(\frac{s}{\epsilon} - 1\big) dB_s, \quad t \geq 0. \tag{4.7}$$

Then $\big(R_\epsilon^1(K,t), t \geq 0\big)$ converges in the ucp sense to 0.

Proof of Lemma 4.4. Since (K_t) is locally bounded there exists a increasing sequence of stopping times $(T_n)_{n\geq 1}$ such that $T_n \uparrow \infty$ as $n \to \infty$ and $|K(t \wedge T_n)| \leq n$, for any $t \geq 0$. Then, according to Lemma 4.3 it is sufficient to prove that $\big(R_\epsilon^1(K,t), t \geq 0\big)$ converges in the ucp sense to 0 when (K_t) is bounded. In that case, using Doob's inequality we get :

$$E\Big(\sup_{t\in[0,T]} \big(R_\epsilon^1(K,t)\big)^2\Big) \leq \frac{C}{\epsilon} E\Big(\int_0^\epsilon K(s)^2 \big(\frac{s}{\epsilon} - 1\big)^2 ds\Big) \leq C \sup_{0\leq s\leq\epsilon} E\big(K(s)^2\big)$$

where $T > 0$.

Recall that (K_s) is bounded, $s \mapsto K(s)$ is right continuous at 0, $K(0) = 0$, then the dominated convergence theorem implies that $\lim_{\epsilon\to 0} \Big(\sup_{0\leq s\leq\epsilon} E\big(K(s)^2\big)\Big) = 0$. This proves that $\sup_{t\in[0,T]} \big|R_\epsilon^1(K,t)\big|$ goes to 0 in $L^2(\Omega)$. ■

Note that under (1.9), relation (2.2) implies that :

$$\Delta_\epsilon(H,t) = \widetilde{\Delta}_\epsilon(H,t) + R_\epsilon^1(H\Phi,t) \tag{4.8}$$

where $\Delta_\epsilon(H,t)$ has been defined by (1.7) and

$$\widetilde{\Delta}_\epsilon(H,t) := \frac{1}{\epsilon\sqrt{\epsilon}}\int_0^t \left(\int_{(u-\epsilon)_+}^u (H_s - H_u)ds\right)\Phi(u)dB_u. \tag{4.9}$$

■

3. Proof of Proposition 1.8. Recall that $\Delta_\epsilon^{(2)}(K,t)$ is defined by (1.11). Using Itô's formula, we obtain:

$$(B_{(s+\epsilon)\wedge t} - B_s)^2 = 2\int_s^{(s+\epsilon)\wedge t}(B_u - B_s)dB_u + (s+\epsilon)\wedge t - s.$$

Reporting in $\Delta_\epsilon^{(2)}(K,t)$ and applying stochastic Fubini's theorem lead to

$$\Delta_\epsilon^{(2)}(K,t) = 2\int_0^t K_u dW_\epsilon(u) + R_\epsilon^1(t) + R_\epsilon^2(t),$$

where

$$\begin{aligned}
R_\epsilon^1(t) &:= \frac{2}{\epsilon\sqrt{\epsilon}}\int_0^t \left[\int_{(u-\epsilon)^+}^u (K_s - K_u)(B_u - B_s)ds\right] dB_u \\
R_\epsilon^2(t) &:= \frac{1}{\epsilon\sqrt{\epsilon}}\int_{(t-\epsilon)^+}^t K_s(t-s-\epsilon)ds.
\end{aligned}$$

Note that Proposition 4.1 (with $\Gamma = K$) ensures the convergence in distribution of $\int_0^\cdot K_u dW_\epsilon(u)$ to $\sigma\int_0^\cdot K_u dW(u)$.

Since $s \to K_s$ is locally bounded, then $\lim_{\epsilon\to 0}\sup_{t\in[0,T]}|R_\epsilon^2(t)| = 0$ a.s.

To prove that $R_\epsilon^1 \overset{(ucp)}{\to} 0$, we may assume that K is bounded (cf Lemma 4.3). Using the Cauchy-Schwarz and Doob inequalities, we obtain successively :

$$\begin{aligned}
E\Big(\sup_{0\le t\le T}(R_\epsilon^1)^2\Big) &\le \frac{C}{\epsilon^3}\int_0^T E\left(\Big(\int_{(u-\epsilon)^+}^u (K_s - K_u)(B_s - B_u)ds\Big)^2\right) du \\
&\le \frac{C}{\epsilon^2}\int_0^T du \int_{(u-\epsilon)^+}^u \sqrt{E\big((K_s-K_u)^4\big)E\big((B_s-B_u)^4\big)}ds \\
&\le C\int_0^T \Big(\sup_{s\le u\le (s+\epsilon)\wedge t} E\big((K_s - K_u)^4\big)\Big)ds
\end{aligned}$$

Since K is bounded and right-continuous, then the term in the right-hand side of the above inequality goes to 0 as $\epsilon \to 0$. ■

4. Proof of Point (1) **of Theorem 1.5.** Using (4.8) we have :

$$\begin{aligned}
\Delta_\epsilon(H_0,t) &= H_0\big(\widetilde{\Delta}_\epsilon(1,t) + R_\epsilon^1(\Phi,t)\big) = H_0 R_\epsilon^1(\Phi,t) \\
&= H_0\Phi(0)N_\epsilon + H_0 R_\epsilon^1\big(\Phi - \Phi(0), t\big)
\end{aligned}$$

where $\epsilon < t$ and

$$N_\epsilon := \frac{1}{\sqrt{\epsilon}} \int_0^\epsilon \big(\frac{u}{\epsilon} - 1\big) dB_u, \quad \epsilon < t.$$

The r.v N_ϵ has a centered Gaussian distribution, with variance

$$E(N_\epsilon^2) = \int_0^\epsilon \Big(\frac{u}{\epsilon} - 1\Big)^2 \frac{du}{\epsilon} = \frac{1}{3} = \sigma.$$

According to Lemma 4.4, $R_\epsilon^1\big(\Phi - \Phi(0), \cdot\big) \overset{(ucp)}{\to} 0$ as $\epsilon \to 0$.

5. Proof of Point (2) **of Theorem 1.5.** Since $(H_t) = (V_t)$ is continuous and $V_0 = 0$ then, Lemma 4.4 applied with $K = \Phi H$ implies that $R_\epsilon^1(\Phi H, \cdot) \overset{(ucp)}{\to} 0$ as $\epsilon \to 0$.

Let $T > 0$. According to Lemmas 4.3 and 2.2, we may suppose that Φ is bounded and :

$$|V_s - V_u| \le C|u - v|^\beta, \quad u, v \in [0, T], \beta > \frac{1}{2}.$$

As a result,

$$\left| \frac{1}{\epsilon\sqrt{\epsilon}} \int_{(u-\epsilon)_+}^u (V_s - V_u) ds \right| \leqslant \frac{1}{\epsilon\sqrt{\epsilon}} \int_{u-\epsilon}^u C\, |s - u|^\beta \, ds \leqslant C\epsilon^{\beta - \frac{1}{2}}$$

and

$$E\Big(\sup_{t\in[0,T]} \big(\widetilde{\Delta}_\epsilon(V, t)\big)^2 \Big) \le C\epsilon^{2\beta - 1}.$$

Using (4.8), item 2 of Theorem 1.5 follows.

6. Proof of Point (3) **of Theorem 1.5.**

a) Recall that $M_t = \displaystyle\int_0^t \Lambda_r dB_r$ and

$$\widetilde{\Delta}_\epsilon(M, t) = -\frac{1}{\epsilon\sqrt{\epsilon}} \int_0^t \left(\int_{(u-\epsilon)^+}^u (M_u - M_s) ds \right) \Phi(u) dB_u.$$

Let $s < u$, we have

$$M_u - M_s = \Lambda_{(u-\epsilon)_+}(B_u - B_s) + \int_s^u \big(\Lambda_r - \Lambda_{(u-\epsilon)_+}\big) dB_r.$$

Using (1.8) we get :

$$\widetilde{\Delta}_\epsilon(M, t) = -\int_0^t \Lambda_{u_-} \Phi(u) dW_\epsilon(u) + R_\epsilon^2(t) + R_\epsilon^3(t), \tag{4.10}$$

where

$$R^2_\epsilon(t) := -\int_0^t \big(\Lambda_{(u-\epsilon)_+} - \Lambda_{u_-}\big)\Phi(u)dW_\epsilon(u)$$

and

$$R^3_\epsilon(t) = -\frac{1}{\epsilon\sqrt{\epsilon}}\int_0^t \left(\int_{(u-\epsilon)_+}^u (r-(u-\epsilon)_+)(\Lambda_r - \Lambda_{(u-\epsilon)_+})dB_r\right)\Phi(u)dB_u.$$

b) Suppose for a while that R^2_ϵ and R^3_ϵ converge in the ucp sense to 0, as $\epsilon \to 0$. Then, Proposition 4.1 with $\Gamma = \Lambda$ implies that the convergence of $\Delta_\epsilon(H,\cdot)$ to $\sigma\int_0^\cdot \Lambda_{u_-}\Phi(u)dW_u$. Note that $\int_0^\cdot \Lambda_u\Phi(u)dW_u = \int_0^\cdot \Lambda_{u_-}\Phi(u)dW_u$ a.s.

c) Let us prove that R^3_ϵ converge in the ucp sense to 0. The proof related to R^2_ϵ is similar and easier; it is left to the reader. From Lemma 4.3, we can suppose that $\big(\Lambda_u, 0 \le u \le T\big)$ and $\big(\Phi(u), 0 \le u \le T\big)$ are bounded. Then, using Burkholder-Davies-Gundy and Hölder inequalities we get :

$$\begin{aligned}
E\Big(\sup_{0\le t\le T} R^3_\epsilon(t)^2\Big) &\le \frac{C}{\epsilon}E\left(\int_0^T \Big\{\int_{(u-\epsilon)_+}^u \frac{r-(u-\epsilon)_+}{\epsilon}(\Lambda_r - \Lambda_{(u-\epsilon)_+})dB_r\Big\}^2\right.\\
&\qquad\qquad \left.\times\Phi(u)^2du\right)\\
&\le \frac{C}{\epsilon}\int_0^T duE\Big(\Big\{\int_{(u-\epsilon)_+}^u \frac{r-(u-\epsilon)_+}{\epsilon}(\Lambda_r - \Lambda_{(u-\epsilon)_+})dB_r\Big\}^2\Big)\\
&\le \frac{C}{\epsilon}\int_0^T du\int_{(u-\epsilon)_+}^u \Big(\frac{r-(u-\epsilon)_+}{\epsilon}\Big)^2 E\big((\Lambda_r - \Lambda_{(u-\epsilon)_+})^2\big)dr\\
&\le C\int_0^T \sup_{(u-\epsilon)_+\le r<u}\Big(E\big((\Lambda_r - \Lambda_{(u-\epsilon)_+})^2\big)\Big)du.
\end{aligned}$$

Using the dominated convergence theorem and the fact that $t \mapsto \Lambda_t$ has left-limits we can conclude that the right-hand side in the above inequality goes to 0 as $\epsilon \to 0$. Consequently, $\sup_{0\le t\le T}|R^3_\epsilon(t)|$ goes to 0 in $L^2(\Omega)$.

■

7. Proof of Proposition 1.6

From (4.8), we have :

$$\Delta_\epsilon\big(h(B),t\big) = \widetilde{\Delta}_\epsilon\big(h(B),t\big) + R^1_\epsilon\big(h(B)\Phi,t\big)$$

where

$$\widetilde{\Delta}_\epsilon\big(h(B),t\big) = \frac{1}{\epsilon\sqrt{\epsilon}}\int_0^t \left(\int_{(u-\epsilon)^+}^u \big\{h(B_s) - h(B_u)\big\}ds\right)\Phi(u)dB_u.$$

Since :

$$\begin{aligned} h(B_s) - h(B_u) &= (B_s - B_u) \int_0^1 h'\big(B_u + \theta(B_s - B_u)\big) d\theta \\ &= (B_s - B_u) h'(B_u) \\ &\quad + (B_s - B_u) \int_0^1 \Big\{ h'\big(B_u + \theta(B_s - B_u)\big) - h'(B_u) \Big\} d\theta \end{aligned}$$

then,

$$\Delta_\epsilon\big(h(B), t\big) = - \int_0^t h'(B_u)\Phi(u) dW_\epsilon(u) + R_\epsilon^1\big(h(B)\Phi, t\big) + R_\epsilon^3(t)$$

where $\big(W_\epsilon(u)\big)$ is the process defined by (1.8) and

$$R_\epsilon^3(t) := \frac{1}{\sqrt{\epsilon}} \int_0^t \left\{ \frac{1}{\epsilon} \int_{(u-\epsilon)_+}^u (B_s - B_u) \Big[\int_0^1 \Big\{ h'\big(B_u + \theta(B_s - B_u)\big) - h'(B_u) \Big\} d\theta \Big] ds \right\} \Phi(u) dB_u.$$

Using Proposition 4.1 (with $\Gamma = h'(B)\Phi$) implies that $\int_0^\cdot h'(B_u)\Phi(u) dW_\epsilon(u)$, converges in distribution to $\sigma \int_0^\cdot h'(B_u)\Phi(u) dW(u)$, as $\epsilon \to 0$. Since $h(0) = 0$, Lemma 4.4 may be applied : $R_\epsilon^1\big(h(B)\Phi, \cdot\big) \overset{(ucp)}{\to} 0$, as $\epsilon \to 0$. We claim that R_ϵ^3 has the same behavior. By localization and Lemma 4.3 we may suppose that Φ is bounded. Using Doob's and Hölder inequalities we obtain :

$$\begin{aligned} E\Big(\sup_{0 \le t \le T} \big(R_\epsilon^3(t)\big)^2 \Big) &\le \frac{C\delta(h', \epsilon)}{\epsilon^2} \int_0^T \left\{ \int_{(u-\epsilon)_+}^u \sqrt{E\big([B_s - B_u]^4\big)} ds \right\} du \\ &\le C\delta(h', \epsilon) \end{aligned}$$

where

$$\delta(\phi, \epsilon) := \sqrt{\sup_{0 \le \theta \le 1, 0 \le u - \epsilon \le s \le u \le T} E\Big(\big\{ \phi\big(B_u + \theta(B_s - B_u)\big) - \phi(B_u) \big\}^4 \Big)}.$$

It can be proved that $\lim_{\epsilon \to 0} \delta(\phi, \epsilon) = 0$ as soon as ϕ is subexponential. As a result, $\sup_{t \le T} \big| R_\epsilon^3(t) \big|$ goes to 0 in $L^2(\Omega)$ as $\epsilon \to 0$.

8. Proof of Theorem 1.7

a) The a.s. converges comes from the continuity of $t \mapsto X_t$ and the identity

$$\frac{1}{\epsilon} \int_0^t H_s\big(X_{s+\epsilon} - X_s\big) ds = \sum_{j=0}^{i-1} h_j \left(\frac{1}{\epsilon} \int_{a_{j+1}}^{a_{j+1}+\epsilon} X_s ds - \frac{1}{\epsilon} \int_{a_j}^{a_j+\epsilon} X_s ds \right)$$

$$+h_i\left(\frac{1}{\epsilon}\int_t^{t+\epsilon} X_s ds - \frac{1}{\epsilon}\int_{a_i}^{a_i+\epsilon} X_s ds\right)$$

where $a_i \le t \le a_{i+1}$ and $i \ge 0$.

b) Let us deal the convergence in distribution. Recall that we supposed that $X = B$. Using the definition of $\Delta_\epsilon(H,t)$, identity (2.2) and easy calculations we get :

$$\Delta_\epsilon(H,t) = h_0\Big\{\Phi(0)G_0(\epsilon) + R^1_\epsilon\big(\Phi - \Phi(0), \epsilon\big)\Big\}, \quad 0 < t \le a_1, 0 < \epsilon < t$$

where $R^1_\epsilon\big(\Phi - \Phi(0), \epsilon\big)$ has been defined by (4.7) and

$$G_0(\epsilon) := \frac{1}{\sqrt{\epsilon}}\int_0^\epsilon \Big(\frac{s}{\epsilon} - 1\Big) dB_s.$$

More generally when $t \in]a_i, a_{i+1}]$, $\epsilon < (t - a_i) \wedge (a_i - a_{i-1})$ and $i \ge 1$, we have

$$\Delta_\epsilon(H,t) = \Delta_\epsilon(H,a_i) + (h_i - h_{i-1})\big(\Phi(a_i)G_i(\epsilon) + \widetilde{R}^1_\epsilon\big)$$

with

$$\begin{aligned} G_i(\epsilon) &:= \frac{1}{\sqrt{\epsilon}}\int_{a_i}^{a_i+\epsilon}\Big(\frac{s-a_i}{\epsilon} - 1\Big) dB_s \\ \widetilde{R}^1_\epsilon &:= \frac{1}{\sqrt{\epsilon}}\int_{a_i}^{a_i+\epsilon}\Big(\frac{s-a_i}{\epsilon} - 1\Big)\big(\Phi(s) - \Phi(a_i)\big) dB_s \end{aligned}$$

As a result for any $t \in]a_i, a_{i+1}]$ we have :

$$\Delta_\epsilon(H,t) = h_0\Phi(0)G_0(\epsilon) + \big(h_1 - h_0\big)\Phi(a_1)G_1(\epsilon) + \cdots + \big(h_i - h_{i-1}\big)\Phi(a_i)G_i(\epsilon) + \big(h_i - h_{i-1}\big)\widetilde{R}^1_\epsilon$$

where

$$\epsilon < (a_1 - a_0) \wedge \cdots \wedge (a_i - a_{i-1}) \wedge (t - a_i). \tag{4.11}$$

Recall that Φ has been supposed to be right-continuous at a_i, then Lemma 4.4 may be applied : $\widetilde{R}^1_\epsilon \stackrel{(ucp)}{\to} 0$, as $\epsilon \to 0$. As a result, the term $\widetilde{R}^1_\epsilon$ gives no contribution to the limit of $\Delta_\epsilon(H,\cdot)$.

Note that $G_i(\epsilon)$ is a Gaussian r.v. with variance $\sigma^2 = 1/3$ and under (4.11) the r.v.'s $G_0(\epsilon), \cdots, G_i(\epsilon)$ are independent and

$$\lim_{\epsilon\to 0} E\big(B_s G_i(\epsilon)\big) = 0, \quad \forall\ s \ge 0.$$

Item 2 of Theorem 1.7 follows.

References

[1] Blandine Bérard Bergery. *Approximation du temps local et intégration par régularisation.* PhD thesis, Nancy Université, 2004-2007.

[2] M. Gradinaru and I. Nourdin. Approximation at first and second order of m-order integrals of the fractional Brownian motion and of certain semimartingales. *Electron. J. Probab.*, 8:no. 18, 26 pp. (electronic), 2003.

[3] J. Jacod, and J. Mémin. Sur un type de convergence intermédiaire entre la convergence en loi et la convergence en probabilité, *in Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980)*, Lecture Notes in Math., 850, 529–546, Springer, 1981.

[4] A. Jakubowski, J. Mémin, and G. Pagès. Convergence en loi des suites d'intégrales stochastiques sur l'espace $\mathbf{D}^1$ de Skorokhod. *Probab. Theory Related Fields*, 81(1):111–137, 1989.

[5] R. Karandikar. On almost sure convergence results in stochastic calculus. In *Séminaire de Probabilités, XXXIX*, Lecture Notes in Math. Springer, Berlin, 1874, 137–147, 2006.

[6] P.A. Meyer. Probabilités et potentiel, *Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. XIV. Actualités Scientifiques et Industrielles*, No. 1318, Hermann, Paris, 1966.

[7] D. Nualart,G. Peccati Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* 33(1): 177–193, 2005.

[8] D. Revuz, and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[9] H. Rootzen. Limit distribution for the error in approximations of stochastic integrals. *Ann. Probab.*, 8(2):241–251, 1980.

[10] F. Russo and P. Vallois. The generalized covariation process and Itô formula. *Stochastic Process. Appl.*, 59(1):81–104, 1995.

[11] F. Russo and P. Vallois. Itô formula for C^1-functions of semimartingales. *Probab. Theory Related Fields*, 104(1):27–41, 1996.

[12] F. Russo and P. Vallois. Stochastic calculus with respect to continuous finite quadratic variation processes. *Stochastics Stochastics Rep.*, 70(1-2):1–40, 2000.

[13] F. Russo and P. Vallois. Elements of stochastic calculus via regularisation. In *Séminaire de Probabilités XL*, Lecture Notes in Math. Springer, Berlin, 1899, 147–185, 2007.